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Compliance Analysis of a Bridged Crack Under Monotonic and Cyclic Loading

Xiaozhi Hu, Yiu-Wing Mai*

Center for Advanced Materials Technology, Department of Mechanical Engineering, University of Sydney, New South Wales 2006, Australia

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Srinivasarao Lathabai

CSIRO, Division of Materials Science & Technology, Clayton, Victoria 3168, Australia

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Abstract

A systematic approach for crack bridging in quasi-brittle materials, such as coarse-grained alumina ceramics, is extended to study frictional degradation in grain bridges during cyclic fatigue. It is shown that the variation in bridging stress owing to the grain-bridge degradation can be determined in terms of compliance measurements. Parallel to the bridging theory under monotonic loading, the extended analysis under cyclic loading is divided into two parts: i.e. the pure frictional degradation of existing bridges without any crack growth, and the coupling problem of the establishment of new bridges accompanying fatigue crack growth and degradation of old bridges subject to cyclic fatigue. Preliminary results on bridging stresses in a coarse-grained alumina under monotonic loading are obtained with the bridging theory.

Es wird eine systematische Annäherung für die Rißüberbrückung in quasi-spröden Materialien, wie beispielsweise grobkörnige Aluminium-Oxid-Keramiken ausgeweitet, um den reibungsbedingten Zerfall in Kornbrücken wahrend zyklischer Ermüdung zu untersuchen. Es wird gezeigt, daß die Variation der Überbrückungsspannung für den Zerfall der Kornbrücken mit Hilfe von Steifigkeitsmessungen bestimmt werden kann. Parallel zu der Überbrückungstheorie für den Fall monotoner Belastung ist die erweiterte Analyse für den Fall der zyklischen Belastung in zwei Teile untergliedert: der rein reibungsbedingte Zerfall bestehender Brücken ohne jegliches Rißwachstum und das Problem der Kopplung der Bildung neuer Brücken in Verbindung mit Ermüdungsrißwachstum und dem Zerfall alter Brücken in Abhängigkeit von der Ermüdung. Es wurden erste Ergebnisse zur Überbrückungsspannung für grobkörniges Aluminium-Oxid unter monotoner Belastung mit Hilfe der Überbrückungstheorie erzielt.

Une approche systématique du phénomène de pontage des fissures dans des matériaux quasi-fragiles, comme des céramiques alumines à gros grain, est étendue afin d'étudier la dégradation par frottement des ponts assurés par les grains lors d'une fatigue cyclique. On a montré que la variation de la contrainte de pontage, imputable à la dégradation de ces ponts, peut être déterminée par des mesures de compliance. Parallèlement à la théorie du pontage sous charge constante, l'analyse étendue à des chargements cycliques est divisée en deux parties: la dégradation par pur frottement des ponts existants, sans aucune croissance de fissures, et les problèmes couplés de la formation de nouveaux ponts accompagnant la croissance de fissures par fatigue et de la dégradation des anciens ponts soumis à une fatigue cyclique. Les résultats préléminaires sur les contraintes de pontage, sous charge constante, dans une alumine à gros grain, sont obtenus par la théorie du pontage.

1 Introduction

It has been shown that crack-interface grain 'bridging' behind a crack tip is the major toughening

* To whom correspondence should be addressed.

mechanism in alumina ceramics.[1-4] Swanson *et al.*[1] have provided the most convincing visual evidence of such crack-interface grain bridges. The evolution of grain bridging and meandering of the fracture path in a coarse-grained alumina are clearly demonstrated in their in-situ photomicrographs. The sawcut experiments performed by Knehans & Steinbrech[4] also confirm the bridging influence on the crack resistance (R) curve. Since the R-curve behaviour of alumina ceramics is dependent on the crack-interface bridging stresses, bridging stress measurements are crucial for a better understanding of the toughness behaviour. A bridging stress theory has recently been developed by Hu & Mai[5] based on a compliance analysis.[6-8] Bridging stresses can be determined with the simplified bridging theory from measurements of unloading compliances as a function of crack growth for monotonically applied stresses. The purpose of this communication is to further discuss how the bridging theory can be extended to study frictional degradation of crack-interface bridges, which has been observed in alumina ceramics due to cyclically applied stresses,[9] and to incorporate this degradation effect in fatigue crack growth.

2 Compliance Analysis

2.1 Crack growth under monotonic loading

A fully saturated bridging zone X is shown schematically in Fig. 1, where a is the crack length, σ_m the maximum bridging stress and w_c the critical crack opening displacement. The magnitude of the bridging stress $\sigma_b(x)$ at position x is controlled by the crack opening displacement w. A power law function is convenient to describe this relationship,[10] i.e.

$$\frac{\sigma_b}{\sigma_m} = \left[1 - \frac{w}{w_c}\right]^n \tag{1}$$

where $n > 0$ as σ_b decreases with w for alumina ceramics.

Let C_u be the experimental unloading compliance of a specimen with a crack length a and a fully saturated bridging zone X. Hence C_u contains the influence of bridging stresses transferred within the bridging zone. If $C(a)$ is the theoretical elastic compliance for the same crack length without the bridges, then $C(a) > C_u$. This inequality indicates that if $C_u = C(a_{eff})$, $a_{eff} < a$. Hence, the true crack length a in an alumina ceramic would be underestimated if it were to be determined by C_u based on the conventional compliance method without due

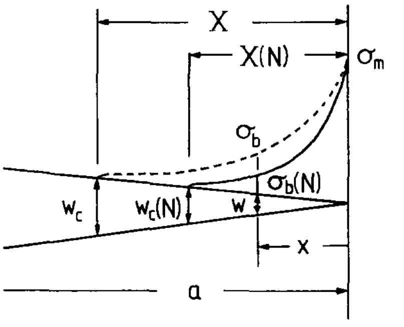

Fig. 1. A fully developed bridging zone X behind the crack tip a, which is reduced to $X(N)$ after N fatigue cycles have elapsed.

consideration of the bridging effect. Therefore, the difference between $C(a)$ and C_u provides an assessment of the bridging stress influence. It should be pointed out that eqn (1) is applicable only for equilibrium stable crack growth. During unloading, the bridging stresses are reduced accordingly. However, they regain their previous values described by eqn (1) after the specimen is reloaded to the equilibrium condition, provided any damage in grain bridges is limited. Hence, C_u can be related to σ_b in eqn (1), although it is determined through an unloading process.

It has been shown in an alumina[11] and a fibre composite[10] that the crack profile within a bridging zone is almost linear. From Fig. 1, $w/w_c = x/X$. And it has been proven[6-8] that

$$\frac{C(a)}{C'(a)}\left\{\frac{C(a)}{C_u} - 1\right\} = X\frac{G_f}{\sigma_m w_c} \tag{2}$$

where $C'(a) = \mathrm{d}C(a)/\mathrm{d}a$ and G_f is the specific fracture energy defined as the area under the $\sigma_b - w$ curve:

$$G_f = \int_0^{w_c} \sigma_b \, \mathrm{d}w = \frac{\sigma_m w_c}{n+1} \tag{3}$$

Let a_0 be the initial notch length. Hence $\Delta a = a - a_0$. An unsaturated bridging zone, i.e. $\Delta a < X$ is similar to a bridging zone which has been fully developed and then partially removed by re-notching. Therefore, the bridging stress theory[6-8] developed for consecutive cutting can be used to determine the bridging stress σ_T at the initial notch tip a_0:

$$\frac{\sigma_T}{\sigma_m} = \frac{\sigma_b}{\sigma_m} = -\frac{C^2(a)}{C'(a)}\frac{C'_u}{C_u^2} \qquad \Delta a < X \tag{4}$$

where $C'(a) = \mathrm{d}C(a)/\mathrm{d}a$, and $C'_u(= \mathrm{d}C_u/\mathrm{d}x)$ is the rate of change in the unloading compliance if the bridging zone is removed by $\mathrm{d}x$ at the initial notch tip. During unloading of a specimen, only C_u, rather than C'_u, is measured. However, with eqn (1), eqn (4) can be integrated over $0 < x < \Delta a$, with $C_u(0) = C(a)$ and $C_u(\Delta a) = C_u$.[5] Thus:

$$\frac{C(a)}{C'(a)}\left\{\frac{C(a)}{C_u} - 1\right\} = \frac{X}{n+1}\left\{1 - \left[1 - \frac{\Delta a}{X}\right]^{n+1}\right\} \quad (5)$$

where C_u is simply the unloading compliance. The left hand side of eqn (2) or (5) is defined as a compliance ϕ-function in Ref. 5. From eqns (2), (3) and (5) it is obtained that

$$\phi(a) = \frac{C(a)}{C'(a)}\left\{\frac{C(a)}{C_u} - 1\right\} = \begin{cases} \dfrac{X}{n+1} & \Delta a \geq X \\ \dfrac{X}{n+1}\left\{1 - \left[1 - \dfrac{\Delta a}{X}\right]^{n+1}\right\} & \Delta a < X \end{cases} \quad (6)$$

With measurements of C_u and a, a ϕ-curve can be constructed with eqn (6), which can be used to determine the softening index n for the bridging stress distribution specified in eqn (1) and the steady-state bridging zone X.

2.2 Frictional degradation under cyclic loading

The ϕ-curve approach given in eqn (6) can be extended to consider bridging stress variations in cyclic fatigue due to frictional degradation of the bridges. Let us consider a specimen with a fully saturated bridging zone X. During cyclic fatigue, the compliance C_u will vary with both the crack length a and the frictional degradation process in the bridging zone. However, the frictional degradation can be separated and evaluated by the compliance ϕ-function if crack growth can be suppressed or limited in length compared to X.

Let N be the number of fatigue cycles elapsed. Since a is approximately constant, only the unloading compliance C_u in the ϕ-function is affected by the frictional degradation. From eqns (2) and (6) it can be shown that

$$\phi(N) = \frac{C(a)}{C'(a)}\left\{\frac{C(a)}{C_u(N)} - 1\right\} = \frac{G_f(N)}{\sigma_m w_c(N)} X(N) = \frac{1}{n+1} X(N) \quad (7)$$

The frictional degradation reduces both the magnitude of the bridging stress $\sigma_b(N)$ and the critical crack opening displacement $w_c(N)$. Hence $G_f(N)$, the area under the $\sigma_b(N) - w_c(N)$ curve (see Fig. 1), is reduced. Since only the bridging stress and not the bridging mechanism is changed during cyclic fatigue, it is reasonable to assume that the softening index n remains constant. Thus, both $w_c(N)$ and $G_f(N)$ are reduced proportionally during cyclic fatigue so that

$$\frac{G_f(N)}{w_c(N)} = \frac{G_f}{w_c} \quad (8)$$

With the above assumption, the change in the compliance $C_u(N)$ or the $\phi(N)$-function after a number of fatigue cycles N has elapsed is directly related to the reduction of the bridging zone X. Let N_c be the critical number of fatigue cycles at which the previously saturated bridging zone is reduced to zero because of the accumulated frictional degradation process between interlocking grains. Thus if $N = N_c$, $X(N) = 0$ and $C_u(N) = C(a)$. Let $\Delta\phi(N) = \phi(a) - \phi(N)$. From eqns (6) and (7):

$$\Delta\phi(N) = \phi(a) - \phi(N) = \frac{C^2(a)}{C'(a)}\left\{\frac{1}{C_u} - \frac{1}{C_u(N)}\right\} = \begin{cases} \dfrac{X - X(N)}{n+1} & N < N_c \\ \dfrac{X}{n+1} & N \geq N_c \end{cases} \quad (9)$$

Comparing eqns (6) and (9), it can be seen that $\phi(a)$ and $\Delta\phi(N)$ are similar toughness curves with the same steady value after $\Delta a \geq X$ or $N \geq N_c$.

The length of the bridging zone $X(N)$ during cyclic fatigue can be determined from either eqn (7) or (9), i.e.

$$X(N) = (n+1)\phi(N) = X - (n+1)\Delta\phi(N) \quad (10)$$

The critical crack opening displacement $w_c(N)$ during cyclic fatigue is given by:

$$w_c(N) = \frac{w_c}{X} X(N) = w_c \frac{(n+1)}{X}\phi(N) = w_c\left[1 - \frac{(n+1)}{X}\Delta\phi(N)\right] \quad (11)$$

The corresponding bridging stress distribution at a critical condition during cyclic fatigue is given by:

$$\sigma_b(N, w) = \sigma_m\left[1 - \frac{w}{w_c(N)}\right]^n \quad (12)$$

The variations in σ_b and w_c are shown in Fig. 1 for such a critical condition.

2.3 Fatigue crack growth

Consider now cyclic fatigue crack growth beyond the critical state shown in Fig. 1 at which $\sigma_b(x = 0) = \sigma_m$ and $w = w_c(N)$ at $x = X(N)$. Cyclic fatigue damage of the bridges will result in the extension of a new bridging zone Δa_{fc}. This extension can occur in a continuous stable fashion or a discontinuous manner depending on the level of applied load $\Delta\sigma_a$ and the number of grain bridges encountered. Insofar as the smearing model is concerned fatigue crack growth is always stable and continuous. Thus, for steady-state crack growth at $\Delta\sigma_a$, eqn (7) can be rewritten as:

$$\phi(N,\Delta a_{fc}) = \frac{C(a+\Delta a_{fc})}{C'(a+\Delta a_{fc})}\left\{\frac{C(a+\Delta a_{fc})}{C_u(N,\Delta a_{fc})}-1\right\} \tag{13a}$$

$$= \frac{G_f(N,\Delta a_{fc})}{\sigma_m w_c(N,\Delta a_{fc})}X(N,\Delta a_{fc})$$

$$= \frac{1}{n+1}X(N,\Delta a_{fc}) \tag{13b}$$

Equation (13b) represents a plateau ϕ-value lower than that indicated by eqn (6) because $X(N,\Delta a_{fc}) < X$, due to the frictional degradation suffered by the bridging grains. Quite clearly, the plateau ϕ-value will also depend on $\Delta\sigma_a$.

Discontinous discrete fatigue crack growth may occur in practice because the bridges are not uniformly distributed in the plane of the crack. Further cycling of the bridged crack at its critical condition, Fig. 2(a), may cause the crack to jump ahead by an amount Δa_{fc}. Fig. 2(b). The maximum fatigue crack growth, Δa_{fc}^m, can be estimated as follows. For simplicity, it is assumed that $\Delta a_{fc} < X(N)$ so that $w_c(N)$ remains constant after the discrete crack growth has taken place. Also, for a fixed linearized crack profile:

$$X(N) = X(N,\Delta a_{fc}) \tag{14}$$

is obtained. From eqns (7), (13) and (14)

$$\alpha = \frac{\phi(N)}{\phi(N,\Delta a_{fc})} = \frac{G_f(N)}{G_f(N,\Delta a_{fc})} \tag{15}$$

Clearly $\alpha \le 1$, but to estimate Δa_{fc}^m the value of α is taken as $\alpha = 1$ and the small part of energy

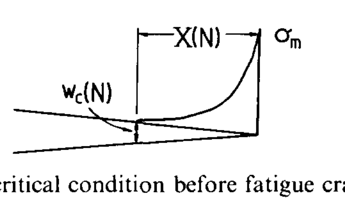

Fig. 2. (a) A critical condition before fatigue crack growth.

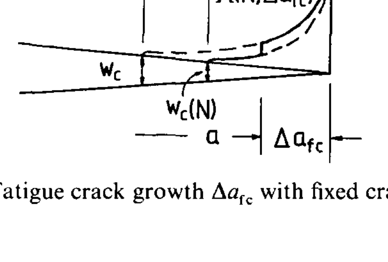

Fig. 2. (b) Fatigue crack growth Δa_{fc} with fixed crack contour.

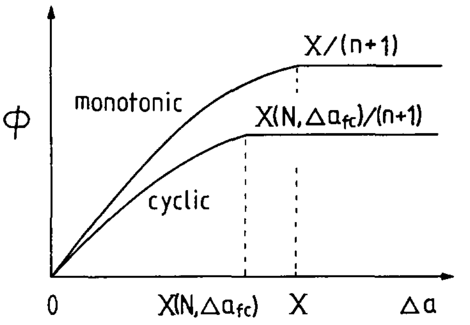

Fig. 3. ϕ-Curves for both monotonic and cyclic loadings.

contribution from the previous bridging stress distribution $\sigma_b(N)$ in Fig. 2(b) is neglected. Thus, by equating $G_f(N)$ to $G_f(N,\Delta a_{fc}^m)$, after simplification:

$$\Delta a_{fc}^m \le X\left\{1-\left[1-\frac{w_c(N)}{w_c}\right]^{1/(n+1)}\right\} \tag{16}$$

is obtained. Equation (16) also suggests that the maximum discrete crack growth Δa_{fc}^m increases with increasing $w_c(N)$ and hence with $\Delta\sigma_a$. Evidence of discontinuous fatigue crack growth in an alumina has recently been observed by Lathabai *et al.* even though this was not reported in Ref. 9.

If continuous fatigue crack growth from an initial notch at a given $\Delta\sigma_a$ is considered the bridged crack will suffer accumulated friction damage as it is prolonged with increasing loading cycles N. A ϕ-curve can be obtained for this situation using the same eqn (13a). Obviously, since $C_u(N,\Delta a_{fc}) > C_u$, the rising ϕ-curve must be less than that for a non-fatigue crack given by eqn (5). Figure 3 shows schematically the ϕ-curves for both monotonic and cyclic loadings. When the fatigue crack achieves its saturated length $X(N)$, ϕ reaches its maximum value obtained from eqn (13b).

3 K_R- and ϕ-Curves of Alumina

In Ref. 9 the K_R-curve of a coarse-grained alumina with an average grain size of 35 μm was measured in a compact tension (CT) specimen, which is shown in Fig. 4. The specimen had an initial a_0/W ratio of 0·325 with $a_0 = 14{\cdot}28$ mm and the thickness was 2 mm. It can be seen from the K_R-curve that at the commencement of crack growth there is a rapid crack extension of about 1 mm although the specimen has been pre-cracked from the sawcut notch. It appears that after a crack extension of $\Delta a > 5$ mm, the plateau K_∞ of about 5·5 MPa$\sqrt{m}$ is reached.

The first ϕ-curve of the alumina calculated from $C(a)$ and C_u with eqn (6) is also given in Fig. 4, which

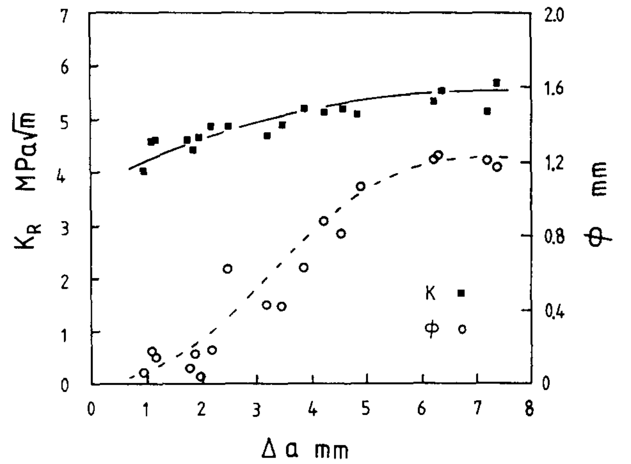

Fig. 4. Toughness curves of an alumina with a mean grain size of 35 μm determined from CT specimens.

clearly shows a steady increase of ϕ with crack growth until the plateau of $\phi_\infty = 1{\cdot}2$ mm is reached for $\Delta a > 5$ mm. It can be proven from eqn (6) that $\phi \approx \Delta a$ at the beginning of the crack propagation for $\Delta a \ll X$. Therefore, the first measurement of ϕ in Fig. 4 with $\Delta a = 1$ mm implies that the crack-interface bridging in the alumina specimen is established at a site about 1 mm away from the initial crack tip. The abrupt crack extension of nearly 1 mm also supports this prediction. Consequently, the fully saturated bridging zone in the specimen should be between 4 and 5 mm, rather than 5 and 6 mm as indicated in Fig. 4. Assuming $X = 4{\cdot}5$ mm, $n = 2{\cdot}8$ is obtained from eqn (6). This value should be compared to those determined with other methods, for instance, $n \approx 2{\cdot}5$ obtained from in-situ scanning electron microscopy (SEM) measurements of the crack profile behind a crack tip for an alumina with an average grain size of 11 μm;[11] $n \approx 3$ determined from R-curve by trial and error[12] for an alumina of 16 μm grain size; and $n = 2{\cdot}1$ estimated from the compliance method for an alumina of 20 μm grain size.[13]

4 Discussion

The first ϕ-curve shown in Fig. 2 provides a good basis for the construction of the second ϕ-curve indicated by eqn (9) for the frictional degradation analysis. As the steady-state value is already known, only a few points are required to give a reasonable estimation of the second ϕ-curve. Experiments are being carried out on an alumina to obtain this ϕ-curve and the results will be reported elsewhere.[14] The simplicity and generality of the ϕ-curve approach in the determination of the bridging stress distribution and frictional degradation are very attractive for the fracture analysis of bridged cracks under both monotonic and cyclic loading conditions. The application of the ϕ-function for fatigue crack growth as indicated by eqns (13) and (16) merits further investigation.

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